

A COMPARISON OF LARGE SCALE DIMENSION OF A METRIC SPACE TO THE DIMENSION OF ITS BOUNDARY

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ABSTRACT. Buyalo and Lebedeva have shown that the asymptotic dimension of a hyperbolic group is equal to the dimension of the group boundary plus one. Among the work presented here is a partial extension of that result to all groups admitting \mathcal{Z} -structures; in particular, we show that $\text{asdim} G \geq \dim Z + 1$ where Z is the \mathcal{Z} -boundary.

1. INTRODUCTION

The primary goal of this paper is to establish a connection between the asymptotic dimension of a group admitting a \mathcal{Z} -structure and the covering dimension of the group's boundary.

For hyperbolic G , the relationship is strong; Buyalo and Lebedeva [5] have shown that $\text{asdim} G = \dim \partial G + 1$. In [6], a partial extension to $\text{CAT}(0)$ groups was attempted. Specifically, it was claimed that $\text{asdim} G \geq \dim \partial G + 1$, where ∂G is any $\text{CAT}(0)$ boundary of G . However, in MathSciNet review MR3058238, X. Xie pointed out a critical error in the proof. Here we recover the same inequality as a special case of a more general theorem.

Theorem 1. *Suppose a group G admits a \mathcal{Z} -structure, (\overline{X}, Z) . Then $\dim Z + 1 \leq \text{asdim} G$.*

By a \mathcal{Z} -structure on G , we are referring to the axiomatized approach to group boundaries laid out in [2] and expanded upon in [8]. Groups known to admit \mathcal{Z} -structures include: hyperbolic groups (with X being a Rips complex and $Z = \partial G$) [3]; $\text{CAT}(0)$ groups (with X being the $\text{CAT}(0)$ space and Z its visual boundary) [2]; systolic groups [15], Baumslag-Solitar groups [10]; as well as various combinations of these classes, as described in [17], [7], and [13]. Definitions of \mathcal{Z} -structure and other key terms used here will be provided in the next section.

Theorem 1 will be obtained from a more general observation about metric spaces.

Work on this project was aided by a Simons Foundation Collaboration Grant.

Theorem 2. *Suppose a proper metric space (X, d) admits a controlled \mathcal{Z} -compactification $\overline{X} = X \cup Z$. Then $\dim Z + 1 \leq \dim_{\text{mc}} X$.*

Here, \dim_{mc} stands for Gromov's *macroscopic dimension*, a type of large scale dimension for metric spaces that is less restrictive than asymptotic dimension in that, for any (X, d) , $\dim_{\text{mc}} X \leq \text{asdim} X$. To complete the proof of Theorem 1 it will then suffice to show that, for a \mathcal{Z} -structure (\overline{X}, Z) on a group G , \overline{X} is a controlled \mathcal{Z} -compactification and $\text{asdim} X = \text{asdim} G$.

Theorem 2 is inspired by the main argument in [11] together with the point of view presented in [14].

2. BACKGROUND AND DEFINITIONS

We begin by providing a few definitions and results for the different dimension theories and then we discuss controlled \mathcal{Z} -compactifications and \mathcal{Z} -structures.

Given a cover \mathcal{U} of a metric space X , $\text{mesh}(\mathcal{U}) = \sup\{\text{diam}(U) | U \in \mathcal{U}\}$. The cover is **uniformly bounded** if there exists some $D > 0$ such that $\text{mesh}(\mathcal{U}) \leq D$. The **order** of \mathcal{U} is the smallest integer n for which each element $x \in X$ is contained in at most n elements of \mathcal{U} .

Definition 3. *The **covering dimension** of a space X is the minimal integer n such that every open cover of X has an open refinement of order at most $n + 1$.*

There are various ways to show that a space has finite covering dimension. When working with compact metric spaces, we prefer the following.

Lemma 4. *For a compact metric space X , $\dim X \leq n$ if and only if, for every $\epsilon > 0$, there is an open cover \mathcal{U} of X with $\text{mesh}\mathcal{U} < \epsilon$ and $\text{order}\mathcal{U} \leq n + 1$.*

Covering dimension can be thought of as a small-scale property. Gromov introduced asymptotic dimension as a large scale analog of covering dimension [9].

Definition 5. *The **asymptotic dimension** of a metric space X is the minimal integer n such that for every uniformly bounded open cover \mathcal{V} of X , there is a uniformly bounded*

open cover \mathcal{U} of X with $\text{order}(\mathcal{U}) \leq n + 1$ so that \mathcal{V} refines \mathcal{U} . In this case, we write $\text{asdim } X = n$.

For a nice survey of asymptotic dimension, see [1]. Although Theorem 1 is stated for asymptotic dimension, we will prove a stronger result using a weaker notion of large scale dimension known as (*Gromov*) *macroscopic dimension*.

Definition 6. *The **Gromov macroscopic dimension** of a metric space X is the minimal integer n such that there exists a uniformly bounded open cover of X with order at most $n + 1$. In this case, we write $\text{dim}_{\text{mc}} X = n$.*

Clearly $\text{dim}_{\text{mc}} X \leq \text{asdim } X$ for every metric space X .

As noted in the introduction, Theorem 1 about groups and their boundaries will be deduced from a broader observation about certain \mathcal{Z} -compactifications of metric spaces. Recall that a closed subset, A , of an ANR, Y , is a **\mathcal{Z} -set** if there exists a homotopy $H : Y \times [0, 1] \rightarrow Y$ such that $H_0 = \text{id}_Y$ and $H_t(X) \subset Y - A$ for every $t > 0$.

Definition 7. *A **controlled \mathcal{Z} -compactification** of a proper metric space X is a compactification $\overline{X} = X \cup Z$ satisfying the following two conditions:*

- *Z is a \mathcal{Z} -set in \overline{X}*
- *For every $\epsilon > 0$ and every $R > 0$, there exists a compact set $K \subset X$ so that every ball of radius R in X not intersecting K has diameter less than ϵ in \overline{X} .*

*In this case, Z is called a **\mathcal{Z} -boundary**, or simply a **boundary** for X .*

There are a few things to take note of in the above definition. First, we have followed tradition and defined \mathcal{Z} -sets in ANRs; hence the compactification \overline{X} must be an ANR. Furthermore, since open subsets of ANRs are also ANRs, X must be an ANR to be a candidate for a controlled \mathcal{Z} -compactification¹. Secondly, it is important to distinguish between the (proper) metric d on X and the metric \bar{d} on \overline{X} . The second condition, which we call the *control condition*, says balls of radius R in (X, d) get arbitrarily small near the boundary, when viewed in (\overline{X}, \bar{d}) . The metric d is crucial; it is given in advance and

¹See Remark 2.

determines the geometry of X . For our purposes the metric on \overline{X} is arbitrary; any \overline{d} determining the appropriate topology can be used.

Example 8. *The addition of the visual boundary to a proper $CAT(0)$ space is a prototypical example in Geometric Group Theory of a controlled \mathbb{Z} -compactification.*

In the presence of nice group actions, controlled Z -compactifications arise rather naturally. As a result, our discussion can be extended to asymptotic dimension of groups and covering dimension of group boundaries. The following definition is key.

Definition 9. *A \mathbb{Z} -structure on a group G is a pair of spaces (\overline{X}, Z) satisfying the following four conditions:*

- (1) \overline{X} is a compact AR,
- (2) Z is a \mathbb{Z} -set in \overline{X} ,
- (3) $X = \overline{X} - Z$ is a proper metric space on which G acts properly, cocompactly, by isometries, and
- (4) \overline{X} satisfies a nullity condition with respect to the action of G on X : for every compact $C \subseteq X$ and any open cover \mathcal{U} of \overline{X} , all but finitely many G translates of C lie in an element of \mathcal{U} .

Remark 1. This definition of \mathbb{Z} -structure is due to Dranishnikov [8]. It generalizes Bestvina's original definition from [2] by allowing \overline{X} to be infinite-dimensional and G to have torsion. We have added an explicit requirement that the metric on X be proper; a quick review of [8] reveals that this requirement was assumed there as well.

3. PROOFS

We begin with a proof of Theorem 2, as the other results will be obtained from it. A key ingredient is the following classical fact about covering dimension.

Lemma 10. [12] *For any nonempty locally compact metric space X , $\dim(X \times [0, 1]) = \dim X + 1$.*

Proof of Theorem 2 . Suppose X admits a controlled \mathcal{Z} -compactification, $\overline{X} = X \cup Z$, and let $\epsilon > 0$. Assume that $\dim_{\text{mc}} X = n$ and let \mathcal{U} of X be a uniformly bounded open cover with $\text{order}(\mathcal{U}) \leq n + 1$.

Using the control condition, we may choose a compact set K_0 such that $\text{diam}_{\overline{d}} U \leq \frac{\epsilon}{3}$ for every $U \in \mathcal{U}$ with the property that $U \cap K_0 = \emptyset$. Let $\mathcal{U}' = \{U \in \mathcal{U} \mid U \cap K_0 = \emptyset\}$.

Since Z is a \mathcal{Z} -set, there is a homotopy $J : \overline{X} \times [0, 1] \rightarrow \overline{X}$ such that $J_0 = \text{id}_{\overline{X}}$ and $J_t(\overline{X}) \cap Z = \emptyset$ for all $t > 0$. By compactness there is some $T > 0$ such that $\overline{d}(z, J_t(z)) < \frac{\epsilon}{3}$ for all $z \in Z$ and $t \in [0, T]$. Furthermore, we may choose $T' > 0$ so that $J(Z \times (0, T']) \subset \bigcup_{U \in \mathcal{U}'} U$. Set $t_0 = \min\{T, T'\}$.

Define $H : \overline{X} \times [0, 1] \rightarrow \overline{X}$ by setting $H(x, t) = J(x, t_0 \cdot t)$. Restrict H to $Z \times [0, 1]$. We will reparametrize $H : Z \times [0, 1] \rightarrow \overline{X}$ in a manner similar to [11], so that pre-images of the open sets in \mathcal{U}' have small mesh. After one additional adjustment, those pre-images will form the desired cover of $Z \times [0, 1]$. For convenience we will use the ℓ_∞ metric on $Z \times [0, 1]$, $d_\infty = \max\{\overline{d}, |\cdot|\}$, where $|\cdot|$ is the standard metric on $[0, 1]$.

Pick $n \in \mathbb{Z}^+$ so that $\frac{3}{n} < \frac{\epsilon}{3}$. Choose $t_1 > t_2 > \dots > t_{n+1} \in [0, 1]$ and compact sets $K_1, K_2, \dots, K_{n+1} \subset X$ as follows:

- let $t_1 = 1$ and choose K_1 so that $H(Z \times \{1\}) \subset K_1$
- for $i = 2, 3, \dots, n$, choose t_i so that $H(Z \times [0, t_i]) \cap K_{i-1} = \emptyset$ and $K_i \subset X$ so that $H(Z \times [t_i, 1]) \cup K_{i-1} \subset K_i$ and K_i contains all elements of \mathcal{U}' that intersect K_{i-1} .
(By properness, elements of \mathcal{U}' have compact closures in X .)
- let $t_{n+1} = 0$ and $K_{n+1} = \overline{X}$.

Let $\lambda : [0, 1] \rightarrow [0, 1]$ be piecewise linear with $\lambda(0) = 0$, $\lambda(1) = 1$, and $\lambda(\frac{i}{n}) = t_{n-i+1}$. Reparametrize H using λ and then push $Z \times [0, 1]$ completely into X by using the map $F : Z \times [0, 1] \rightarrow X$ defined by $F(z, s) = H(z, \lambda(s))$ for $s \in [\frac{1}{n}, 1]$ and $F(z, s) = H(z, \frac{1}{n})$ for $s \in [0, \frac{1}{n}]$.

We show that $\mathcal{V} = \{F^{-1}(U) \mid U \in \mathcal{U}'\}$ is an open cover of $Z \times [0, 1]$ with mesh at most ϵ and order at most $n + 1$.

Let $(z, s), (z', s') \in F^{-1}(U)$ and set $y = F(z, s), y' = F(z', s')$ and $t = \lambda(s), t' = \lambda(s')$. Choose $j \in \{1, 2, \dots, n + 1\}$ such that $y \in K_j - K_{j-1}$. By the choice of K_i and t_i 's above, $t_{j+1} < t < t_{j-1}$. Thus, $\frac{n-j}{n} < s < \frac{2+n-j}{n}$. Since $y, y' \in U$ and $y \in K_j$, then $U \cap K_j = \emptyset$,

so $y' \in K_{j+1}$. Furthermore, $y' \notin K_{j-2}$ because if it were, $U \cap K_{j-2} \neq \emptyset$ and $U \subset K_{j-1}$, a contradiction to the choice of j . Thus, $y' \in K_{j+1} - K_{j-2}$. Similar reasoning as above for t shows that $t_{j+2} < t' < t_{j-2}$ and $\frac{n-1-j}{n} < s' < \frac{n+3-j}{n}$. Thus,

$$|s - s'| < \frac{n+3-j}{n} - \frac{n-j}{n} = \frac{3}{n} < \epsilon$$

Moreover,

$$\begin{aligned} \bar{d}(z, z') &\leq \bar{d}(z, y) + \bar{d}(y, y') + \bar{d}(y', z') \\ &= \bar{d}(z, H(z, \lambda(s))) + \bar{d}(y, y') + \bar{d}(z' H(z', \lambda(s'))) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

By the above $d_\infty((z, s), (z', s')) < \epsilon$, proving $\text{mesh}_{d_\infty} \mathcal{V} < \epsilon$. Since \mathcal{V} consists of the pre-images of \mathcal{U}' and $\text{order} \mathcal{U}' \leq n+1$, then $\text{order} \mathcal{V} \leq n+1$. Using the definition of dimension in Lemma 4 we have $\dim(Z \times [0, 1]) \leq n$ and an application of Lemma 10 finishes the claim. \square

Remark 2. We have chosen to follow the traditional definition of \mathcal{Z} -sets and require \bar{X} to be an ANR. However, the above proof also applies to more general metric spaces. In particular, we make no use of the ANR properties of \bar{X} or X ; if Z is a closed subset of any compact metric space \bar{X} and it is possible to instantly homotope \bar{X} off of Z , then the proof of Theorem 2 will go through as above.

From Theorem 2 we obtain a correct proof of the main assertion of [6, Cor.1.2], which does not involve groups.

Corollary 11. *If X is a proper $CAT(0)$ space, then $\text{asdim } X \geq \dim \partial X + 1$.*

To obtain Theorem 1, we first must show that the notion of controlled \mathcal{Z} -compactification applies to a \mathcal{Z} -structure (\bar{X}, Z) on a group G . Since $Z \subseteq \bar{X}$ is a \mathcal{Z} -set, all that remains to show is that open balls in X become small near the boundary. The cocompact action by isometries combined with the nullity condition will grant that control.

Lemma 12. *Suppose a group G admits a \mathcal{Z} -structure, (\bar{X}, Z) . Then \bar{X} is a controlled \mathcal{Z} -compactification of $X = \bar{X} - Z$.*

Proof. Let $\epsilon > 0$ and $R > 0$. By cocompactness, there is a compact set $C \subset X$ such that $X \subset GC$. Choose $d > 0$ and $x_0 \in X$ such that $C \subset B(x_0, d)$. By the nullity condition, there is a compact set $K' \subset X$ such that whenever $gB(x_0, d+R) \cap K' = \emptyset$ for some $g \in G$, then $\text{diam}_{\bar{d}} gB(x_0, d+R) < \epsilon$. Let $K = \overline{N_{2d+R}(K')}$ be the closed $2d+R$ neighborhood of K' in X . We show this is the desired compact set. Thus, let $B(x, R) \subset X$ for some $x \in X$ with $B(x, R) \cap K = \emptyset$. Choose $g \in G$ such that $gx \in C$. Then, $B(x, R) \subset g^{-1}B(x_0, d+R)$ since for any $y \in B(x, R)$,

$$d(y, g^{-1}x_0) \leq d(y, x) + d(g^{-1}x, x_0) < R + d$$

Furthermore, $g^{-1}B(x_0, d+R) \cap K' = \emptyset$. Otherwise, there would be some $z \in g^{-1}B(x_0, d+R) \cap K'$ and $d(x, z) \leq d(x, g^{-1}x_0) + d(g^{-1}x_0, z) < 2d + R$. However, $B(x, R) \cap K = \emptyset$, so, $d(x, K') > 2d + R$. Because $z \in K'$, we obtain the required contradiction.

Thus $\text{diam}_{\bar{d}} g^{-1}B(x_0, d+R) < \epsilon$. $B(x, R)$, being a subset of $g^{-1}B(x_0, d+R)$, will also have diameter smaller than ϵ . \square

Proof of Theorem 1. Suppose a group G admits a \mathcal{Z} -structure (\bar{X}, Z) . By Lemma 12, \bar{X} is a controlled \mathcal{Z} -compactification of X . Thus, by Theorem 2, $\text{asdim} X \geq \dim Z + 1$. Since G acts geometrically on X , G is coarsely equivalent to X (see Corollary 0.9 in [4]). Moreover, by [16], asymptotic dimension is a coarse invariant; so $\text{asdim} X = \text{asdim} G$. \square

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